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Overlapping Approximation of the Exponential Function on the Positive Axis

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We investigate properties of the exponential function concerning the overlapping approximation which was introduced in former papers. We give bounds for the rate of convergence of the sequence of least deviations and give an exact formula for the convergence speed in the case of only one knot on the positive axis.

0. INTRODUCTION

The differential equation $y' + Ay = 0$, where A is a positive definite and Hermitian matrix, and its solution $y(t) = e^{-At} \cdot y_0$, $y_0 \in \mathbb{C}^m$, leads to the problem of how to compute the matrix e^{-At} efficiently. Starting at this point, in [4, 5] we developed the so-called overlapping approximation which deals with a more general problem: given any continuous function f: $[0, \infty) \to \mathbb{R}$, the matrix function $f(At)$ must be approximated by matrix-valued mappings of a simple structure. For this purpose we take a fixed $k \in \mathbb{N}$ and $0 < m < M$ such that the spectrum of A is contained in $[m, M]$. With $\rho := M/m > 1$ and arbitrary $\eta \geqslant 0$ we define

$$\begin{aligned} J_1(\eta) &:= [0, \eta/M], \\ J_\nu(\eta) &:= [\eta\rho^{\nu-2}/M, \eta\rho^{\nu-1}/M] \qquad \text{for} \quad 2 \leqslant \nu \leqslant k, \\ J_{k+1}(\eta) &:= [\eta\rho^{k-1}/M, \infty). \end{aligned}$$

Then for any real polynomial p and $1 \leqslant \nu \leqslant k+1$ it is easy to see that

$$\sup_{t \in J_\nu(\eta)} \|f(At) - p(At)\|_2 \leqslant \|f - p\|_{I_\nu(\eta)},$$

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where $\|\cdot\|_2$ and $\|\cdot\|_I$ denote the spectral norm and the maximum norm, respectively, and $I_\nu(\eta)$ is defined by

$$\begin{aligned} I_1(\eta) &:= [0, \eta], \\ I_\nu(\eta) &:= [\eta\rho^{\nu-3}, \eta\rho^{\nu-1}] \qquad \text{for} \quad 2 \leqslant \nu \leqslant k, \\ I_{k+1}(\eta) &:= [\eta\rho^{k-2}, \infty). \end{aligned}$$

Therefore, a polynomial approximation p for f on $I_\nu(\eta)$ leads to an approximating matrix function $p(At)$ for $f(At)$ on $J_\nu(\eta)$ and we can define the following least deviation, where $E_n(f, I)$ denotes the least deviation of f on I with respect to Π_n.

0.1. DEFINITION. Let $f \in C[0, \infty)$, $k \in \mathbb{N}$, $n \in \mathbb{N}_0$, and $\rho > 1$ be given.

(i) For $\eta \geqslant 0$ set

$$d_{\nu,\rho}^{(n)}(f, \eta) = E_n(f, I_\nu(\eta)) \qquad \text{if} \quad 1 \leqslant \nu \leqslant k$$

and

$$d_{k+1,\rho}^{(n)}(f, \eta) = d_{k+1,\rho}(f, \eta) = E_0(f, I_{k+1}(\eta)).$$

Moreover, set

$$a_{k,\rho}^{(n)}(f, \eta) = \max_{1 \leqslant \nu \leqslant k+1} d_{\nu,\rho}^{(n)}(f, \eta).$$

(ii) The least deviation of f on $[0, \infty)$ with respect to ρ, k, and Π_n is defined as

$$a_{k,\rho}^{(n)}(f) = \inf_{\eta \geqslant 0} a_{k,\rho}^{(n)}(f, \eta).$$

We showed in [5] that for each $\rho > 1$ there is a $q > 1$ such that for each continuous function f the inequality $a_{k,\rho}^{(n)}(f) \geqslant q^{-n}$ holds if n is large enough. Therefore, it is of considerable interest to study functions which possess this best possible convergence speed.

0.2. DEFINITION. We call f geometric w.r.t. k and ρ if for $\alpha_{k,\rho}^{(n)}(f) = (a_{k,\rho}^{(n)}(f))^{1/n}$ the inequality

$$\limsup_{n \to \infty} \alpha_{k,\rho}^{(n)}(f) = 1/q_{k,\rho}(f) < 1$$

holds; i.e., the sequence of least deviations converges to zero geometrically.

In [5] the class of geometric functions was completely characterized. In particular, we showed that this property does not depend on k and ρ and that each geometric function is the restriction of an entire function of finite order.

In this paper we investigate the speed of convergence of the least deviations for the standard geometric function: the exponential function $f(t) = \exp(-t)$. The following remark shows that f is geometric and gives a crude quantitave estimate of the convergence speed by using Taylor series.

0.3. *Remark.* We set $S_n(x) = \sum_{\nu=0}^{n} (-1)^\nu (x^\nu/\nu!)$, choose $\delta \in (0, 1/e)$ satisfying $\exp(\delta/\rho) = 1/(e\delta)$, and set $\eta_n = \delta \cdot n$. Then for $x \in [0, \eta_n]$, there is $\xi \in [0, \eta_n]$ with

$$\begin{aligned} |\exp(-x) - S_n(x)| &= \left| (-1)^{n+1} \frac{\exp(-\xi)}{(n+1)!} x^{n+1} \right| \\ &\leqslant \delta^{n+1} \frac{n^{n+1}}{(n+1)!} \\ &\leqslant \delta^{n+1} \frac{n}{n+1} \exp(n) \frac{1}{\sqrt{2\pi n}} < \delta \cdot (\delta \cdot e)^n. \end{aligned}$$

Moreover, we have $d_{2,\rho}^{(n)}(f, \eta_n) = \frac{1}{2} \exp(-\eta_n/\rho) = \frac{1}{2}(\delta \cdot e)^n$, and because of $\delta \cdot e < 1$ we obtain geometric convergence by using the Taylor expansion of $\exp(-x)$.

In the following, we derive sharper bounds for the convergence speed $1/q_{k,\rho}(f)$. In Section 1 we show that for sufficiently large k the inequality $1/q_{k,\rho}(f) \leqslant (\rho - 1)/(\rho + 1)$ holds, whereas in Section 2 we develop a method to compute the exact value of $q_{1,\rho}(f)$ as a root of a certain equality.

1. Rate of Convergence for Large k

In [4, Korollar 2.3.7] we showed that for fixed k and ρ the convergence speed of $f(t) = \exp(-t)$ satisfies

$$1/q_{k,\rho}(f) \geqslant M(\sqrt{\rho})^{-\sqrt{\rho}/(\sqrt{\rho}-1)} > 0,$$

where $M(x) = (\sqrt{x} + 1)/(\sqrt{x} - 1))$. In order to get upper bounds less than 1, we must consider the behaviour of $\exp(-z)$ on certain ellipses. For a compact interval $I = [a, b]$ and $q > 1$, we define $\mathscr{E}_q(I)$ by

$$x + iy \in \mathscr{E}_q(I) \qquad \text{if} \quad \frac{(2x - a - b)^2}{(\frac{1}{2}(b-a)(q + 1/q))^2} + \frac{(2y)^2}{(\frac{1}{2}(b-a)(q - 1/q))^2} \leqslant 1.$$

Then from [2, Theorem 73] it easily follows that

$$E_n(f, I) \leqslant \frac{2}{r^n(r-1)} \cdot \max_{z \in \mathscr{E}_r(I)} |f(z)| \tag{$*$}$$

for an entire function f and $r > 1$.

1.1. THEOREM. *For $f(t) = \exp(-t)$ and given $\rho > 1$ there is $k_\rho \in \mathbb{N}$ such that $1/q_{k,\rho}(f) \leqslant (\rho - 1)/(\rho + 1)$ for $k \geqslant k_\rho$.*

Proof. We assume that $k > 1$. For $\eta > 0$ and $r_1 > 1$ it follows from $(*)$ that

$$\begin{aligned} d_{1,\rho}^{(n)}(f, \eta) &\leqslant \frac{2}{r_1^n(r_1 - 1)} \cdot \max_{z \in \mathscr{E}_{r_1}(I_1(\eta))} |\exp(-z)| \\ &= \frac{2}{r_1^n(r_1-1)} \cdot \exp(-\eta/2 + \eta(r_1 + 1/r_1)/4), \end{aligned}$$

because the largest value of $|\exp(-z)|$ is reached at the left vertex of the ellipse. Analogously, for $2 \leqslant \nu \leqslant k$ we have

$$\begin{aligned} d_{\nu,\rho}^{(n)}(f, \eta) \leqslant \frac{2}{r_\nu^n(r_\nu - 1)} &\cdot \exp\left(-\frac{\eta}{2}\rho^{\nu-2}\left(\rho + \frac{1}{\rho}\right)\right) \\ &\cdot \exp\left(\frac{\eta}{4}\rho^{\nu-2}\left(\rho - \frac{1}{\rho}\right)\left(r_\nu + \frac{1}{r_\nu}\right)\right) \end{aligned}$$

and

$$d_{k+1,\rho}^{(n)}(f, \eta) = \tfrac{1}{2}\exp(-\eta\rho^{k-2}).$$

Setting $A_1 = \eta/4 > 0$ and $A_\nu = \eta\rho^{\nu-2}(\rho - 1/\rho)/4 > 0$ for $2 \leqslant \nu \leqslant k$, the parts of these upper bounds being relevant w.r.t. r_ν can be written as

$$\frac{\exp(A_\nu(r_\nu + 1/r_\nu))}{r_\nu^n(r_\nu - 1)}.$$

Minimizing the function $g(r) = r^{-n} \cdot \exp(A_\nu(r + 1/r))$ we obtain

$$r_\nu = \frac{n}{2A_\nu} + \sqrt{\left(\frac{n}{2A_\nu}\right)^2 + 1} > 1$$

and with $\eta = n \cdot \delta$ and

$$F(\delta):=\frac{\delta}{2}\frac{\exp(\sqrt{1+\delta^2/4}-\delta/2)}{\sqrt{1+\delta^2/4}+1},$$

$$X_\nu(\delta):=\frac{\delta(\rho^{\nu-1}-\rho^{\nu-3})}{2}$$

$$\cdot\frac{\exp(\sqrt{1+\delta^2(\rho^{\nu-1}-\rho^{\nu-3})^2/4}-\delta(\rho^{\nu-1}+\rho^{\nu-3})/2)}{\sqrt{1+\delta^2(\rho^{\nu-1}-\rho^{\nu-3})^2/4}+1}, \qquad \nu=2,...,k$$

we have

$$d_{1,\rho}^{(n)}(f,\eta)\leqslant\frac{2}{r_1-1}(F(\delta))^n,$$

$$d_{\nu,\rho}^{(n)}(f,\eta)\leqslant\frac{2}{r_\nu-1}(X_\nu(\delta))^n, \qquad 2\leqslant\nu\leqslant k,$$

$$d_{k+1,\rho}^{(n)}(f,\eta)=\tfrac{1}{2}\exp(-\delta\rho^{k-2}n)$$

for arbitrary $\delta>0$. Since $(\delta/2)(\rho^{\nu-1}+\rho^{\nu-3})=((\rho^2+1)/(\rho^2-1))\cdot\delta/2\cdot(\rho^{\nu-1}-\rho^{\nu-3})$, we can simplify the formula for $X_\nu(\delta)$, $2\leqslant\nu\leqslant k$, by setting $\xi=(\delta/2)(\rho^{\nu-1}-\rho^{\nu-3})$ and obtain

$$X_\nu(\delta)=\xi\cdot\frac{\exp(\sqrt{1+\xi^2}-((\rho^2+1)/(\rho^2-1))\,\xi)}{\sqrt{1+\xi^2}-1}=:Y(\xi).$$

Using standard arguments, it is easy to see that Y reaches its maximum at the point $\xi_0=\frac{1}{2}(\rho-1/\rho)>0$, where $Y(\xi_0)=(\rho-1)/(\rho+1)$, which implies that $X_\nu(\delta)\leqslant(\rho-1)/(\rho+1)$ for $\delta>0$. Therefore,

$$\frac{1}{q_{k,\rho}(f)}\leqslant\max\left(F(\delta),\frac{\rho-1}{\rho+1},\exp(-\delta\rho^{k-2})\right) \qquad \text{for} \quad \delta>0.$$

Since $F(0)=0$ we can choose $\delta>0$ such that $F(\delta)<(\rho-1)/(\rho+1)$ and $k_\rho\in\mathbb{N}$ satisfying $\exp(-\delta\rho^{k_\rho-2})<(\rho-1)/(\rho+1)$, but this implies $1/q_{k,\rho}(f)\leqslant(\rho-1)/(\rho+1)$ for $k\geqslant k_\rho$. ∎

2. The Case $k=1$

In the following, we set $q_\rho=q_{1,\rho}(f)$ for $f(t)=\exp(-t)$ and $\rho>1$. Although it was shown in [5, Satz 3] that $1/q_\rho\geqslant 1/M(\rho)$, this lower bound tends to zero for $\rho\to 1$ and therefore cannot be sharp. We show here that q_ρ can be determined as a root of an equation. Since this is an implicit description of q_ρ, we also give a lower bound for q_ρ that can be computed easily.

2.1. THEOREM. *$q_\rho > 1$ is the unique root of the equation $1/q = F(\rho \log q)$, where F is given by*

$$F(\delta) := \frac{\delta}{2} \frac{\exp(\sqrt{1+\delta^2/4} - \delta/2}{\sqrt{1+\delta^2/4}+1}.$$

Proof. We denote by T_ν the νth Tchebyshev polynomial of the first kind and by

$$I_\nu(z) = \sum_{\mu=0}^{\infty} \frac{(z/2)^{2\mu+\nu}}{\mu! \cdot (\nu+\mu)!}$$

the Bessel function of order ν with purely imaginary argument. Then it is well-known that for $t \in \mathbb{R}$ and $z \in \mathbb{C}$

$$\exp(tz) = I_0(t) + 2 \cdot \sum_{\nu=1}^{\infty} I_\nu(t)\, T_\nu(z).$$

We choose an arbitrary $\eta > 0$. If $d_{1,\rho}^{(n)}(f, \eta) = d_{2,\rho}^{(n)}(f, \eta)$ we set $\eta_n = \eta$. If $d_{1,\rho}^{(n)}(f, \eta) > d_{2,\rho}^{(n)}(f, \eta)$, by continuity arguments there is $0 < \eta_n < \eta$ with $d_{1,\rho}^{(n)}(f, \eta_n) = d_{2,\rho}^{(n)}(f, \eta_n)$ and in the case $d_{1,\rho}^{(n)}(f, \eta) < d_{2,\rho}^{(n)}(f, \eta)$ we get $\eta_n > \eta$ with this property. In any case we have

$$\begin{aligned} a_{1,\rho}(f) &= E_n(f, [0, \eta_n]) = E_0(f, [\eta_n/\rho, \infty)) \\ &= \frac{1}{2} \exp\left(\frac{-\eta_n}{\rho}\right) = \frac{1}{q_n^n} \end{aligned}$$

with a suitable $q_n > 1$. Then by [2, Theorem 66] we have

$$\begin{aligned} \frac{1}{q_n^n} &= \exp\left(\frac{-\eta_n}{2}\right) \cdot E_n\left(\exp\left(\frac{\eta_n t}{2}\right), [-1, 1]\right) \\ &\geqslant 2\exp\left(\frac{-\eta_n}{2}\right) \cdot \sum_{\mu=0}^{\infty} I_{(2\mu+1)(n+1)}\left(\frac{\eta_n}{2}\right) \\ &= 2\exp\left(-\frac{\rho n}{2} \log \frac{q_n}{2^{1/n}}\right) \sum_{\mu=0}^{\infty} I_{(2\mu+1)(n+1)}\left(\frac{\rho n}{2} \log \frac{q_n}{2^{1/n}}\right) \\ &> 2\exp\left(-\frac{\rho n}{2} \log \frac{q_n}{2^{1/n}}\right) I_{n+1}\left(\frac{\rho(n+1)}{2} \log \left(\frac{q_n}{2^{1/n}}\right)^{n/n+1}\right). \end{aligned}$$

If we choose an arbitrary $\tilde{q} < q_\rho$, then for sufficiently large n we have $1/q_n \leqslant 1/\tilde{q}$ and $(q_n/2^{1/n})^{n/n+1} \geqslant \tilde{q}$ and, therefore,

$$\frac{1}{\tilde{q}} \geqslant \exp\left(-\frac{\rho}{2}\log\frac{q_n}{2^{1/n}}\right)\left(I_{n+1}\left(\frac{\rho(n+1)}{2}\log\tilde{q}\right)\right)^{1/n}.$$

Obviously, we can substitute $\tilde{q}$ by q_ρ and by choosing a suitable subsequence one gets

$$\frac{1}{q_\rho} \geqslant \exp\left(-\frac{\rho}{2}\log q_\rho\right)\cdot\limsup_{n\to\infty}\left(I_{n+1}\left(\frac{\rho(n+1)}{2}\log q_\rho\right)\right)^{1/n}.$$

Since it is well-known (cf. [3, 7.07 and 7.16] or [1, 9.7.7 and 9.7.11]) that for $t > 0$

$$\lim_{n\to\infty}(I_n(tn))^{1/n} = \frac{t}{1+\sqrt{1+t^2}}\cdot\exp(\sqrt{1+t^2}),$$

we obtain

$$\frac{1}{q_\rho} \geqslant \frac{\rho\log q_\rho}{2}\cdot\frac{\exp(\sqrt{1+\rho^2(\log q_\rho)^2/4} - (\rho\log q_\rho)/2)}{1+\sqrt{1+\rho^2(\log q_\rho)^2/4}}$$
$$= F(\rho\log q_\rho).$$

Following the proof of Theorem 1.1, we also have $1/q_\rho \leqslant \max(F(\delta), \exp(-\delta/\rho))$ for arbitrary $\delta > 0$. Since F is a monotonic increasing function satisfying $F(0) = 0$, there is a unique $\delta_\rho > 0$ such that $F(\delta_\rho) = \exp(-\delta_\rho/\rho)$ and hence

$$F(\delta_\rho) \geqslant \frac{1}{q_\rho} \geqslant F(\rho\log q_\rho).$$

Therefore, $\rho\log q_\rho \leqslant \delta_\rho$, but we also have

$$\exp\left(-\rho\cdot\frac{\log q_\rho}{\rho}\right) = \frac{1}{q_\rho} \leqslant \exp\left(\frac{-\delta_\rho}{\rho}\right),$$

and since $\exp(-t/\rho)$ is decreasing, one gets $\rho\log q_\rho \geqslant \delta_\rho$ and therefore

$$\frac{1}{q_\rho} = F(\rho\log q_\rho). \quad \blacksquare$$

Table 1 gives the values of q_ρ for $\rho = 1, ..., 10$, where q_1 must be understood as $\lim_{\rho\to 1} q_\rho$.

TABLE 1

Convergence Parameter q_ρ and Lower Bounds

ρ	q_ρ	$\sqrt{M(\rho+1)}$
1	2.448526	2.414214
2	1.940108	1.931852
3	1.735422	1.732051
4	1.619778	1.618034
5	1.543693	1.542659
6	1.489042	1.488372
7	1.447472	1.447009
8	1.414548	1.414214
9	1.387677	1.387426
10	1.365231	1.365037

Although Theorem 1.1 makes it possible to compute q_ρ directly (e.g., with the help of Newton's method), it is an implicit formula. Therefore, we give a different method to compute q_ρ as the maximum value of a certain positive function. For this purpose we set

$$h(m) := \left(1 + \frac{2}{\rho}(m-1)\left(1 + \sqrt{1 + \frac{\rho}{m-1}}\right)\right)^{1/m}.$$

2.2. THEOREM. $q_\rho = \max_{m \geqslant 1} h(m)$.

Proof. At first we show that $q_\rho \geqslant h(m)$ for each $m \geqslant 1$ holds. We choose η_n and q_n as in the proof of Theorem 2.1. Then

$$\eta_n = \rho \cdot (n \log q_n - \log 2)$$

and for arbitrary $s > 1$ it follows that

$$\begin{aligned}
\frac{1}{q_n^n} &= \exp\left(\frac{-\eta_n}{2} \cdot E_n\left(\exp\left(\frac{\eta_n t}{2}\right), [-1, 1]\right)\right) \\
&\leqslant \exp\left(\frac{-\eta_n}{2}\right) \cdot \frac{2}{s^n(s-1)} \cdot \max_{z \in \mathscr{E}_s([-1,1])} \left|\exp\left(\frac{-\eta_n z}{2}\right)\right| \\
&= \frac{2}{s^n(s-1)} \cdot \exp(\eta_n(\mu(s) - 1)) \\
&= \frac{2}{s^n(s-1)} \cdot q_n^{n\rho(\mu(s)-1)} \cdot 2^{-\rho(\mu(s)-1)},
\end{aligned}$$

where $\mu(s)=\frac{1}{2}(1+\frac{1}{2}(s+1/s))$. By taking this inequality to the power of $1/n$ and choosing a suitable subsequence one obtains $q_\rho^{\rho(\mu(s)-1)+1}\geqslant s$. Thus for $s=q_\rho^m>1$ we have $\rho\cdot(\mu(q_\rho^m)-1)+1\geqslant m$ and by an easy calculation we get $q_\rho\geqslant h(m)$. Since $1=\lim_{m\to 1}h(m)=\lim_{m\to\infty}h(m)$, there is $m>1$ such that $h(m)=\max_{x\geqslant 1}h(x)$ and $h'(m)=0$. To calculate the derivation of h we write

$$\begin{aligned}
h(m)&=\left(1+\frac{2}{\rho}(m-1)\left(1+\sqrt{1+\frac{\rho}{m-1}}\right)\right)^{1/m}\\
&=\left(1+2\cdot\frac{\sqrt{1+\rho/(m-1)}+1}{(\sqrt{1+\rho/(m-1)})^2-1}\right)^{1/m}\\
&=\left(1+\frac{2}{\sqrt{1+\rho/(m-1)}-1}\right)^{1/m}\\
&=\exp\left(\frac{1}{m}\log\left(1+\frac{2}{\sqrt{1+\rho/(m-1)}-1}\right)\right),
\end{aligned}$$

and by an easy calculation we obtain

$$\begin{aligned}
h'(m)=h(m)\cdot\Bigg(&\frac{1}{m(m-1)\sqrt{1+\rho/(m-1)}}\\
&-\frac{1}{m^2}\log\left(1+\frac{2}{\sqrt{1+\rho/(m-1)}-1}\right)\Bigg)\\
=\frac{h(m)}{m}\cdot&\left(\frac{1}{(m-1)\sqrt{1+\rho/(m-1)}}-\log h(m)\right).
\end{aligned}$$

Since $h'(m)=0$ this leads to $\log h(m)=1/(m-1)\sqrt{1+\rho/(m-1)}$. In order to substitute $(m-1)/\rho$ and $\rho/(m-1)$ in the definition of $h(m)$, we note the equations

$$(m-1)^2+\rho(m-1)-\frac{1}{(\log h(m))^2}=0,$$

$$\frac{1}{(m-1)^2}-\rho(\log h(m))^2\cdot\frac{1}{m-1}-(\log h(m))^2=0$$

and get

$$\frac{m-1}{\rho}=-\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{1}{\delta^2}}\qquad\text{and}\qquad\frac{\rho}{m-1}=\frac{\delta^2}{2}+\delta\cdot\sqrt{1+\frac{\delta^2}{4}},$$

where $\delta = \rho \cdot \log h(m)$. By definition of δ we have $\sqrt{1+\rho/(m-1)} = \delta/2 + \sqrt{1+\delta^2/4}$ and therefore the equation

$$h(m)^m = 1 + \left(-1 + \sqrt{1+\frac{4}{\delta^2}}\right) \cdot \left(1 + \frac{\delta}{2} + \sqrt{1+\frac{\delta^2}{4}}\right) = \frac{2}{\delta} \cdot \left(1 + \sqrt{1+\frac{\delta^2}{4}}\right),$$

which implies that $h(m)^{-m} = (\delta/2)/(1+\sqrt{1+\delta^2/4})$. Thus we can write

$$\begin{aligned}\frac{1}{h(m)} &= h(m)^{-m} \cdot h(m)^{m-1} \\ &= \frac{\delta/2}{1+\sqrt{1+\delta^2/4}} \cdot \exp((m-1)\log h(m)) \\ &= \frac{\delta/2}{1+\sqrt{1+\delta^2/4}} \exp\left(\frac{1}{\sqrt{1+\rho/(m-1)}}\right) \\ &= \frac{\delta/2}{1+\sqrt{1+\delta^2/4}} \exp\left(\sqrt{1+\frac{\delta^2}{4}} - \frac{\delta}{2}\right),\end{aligned}$$

and this means that $h(m)$ is the root of the equation $1/q = F(\rho \log q)$. Using Theorem 2.1. we get $\max_{x \geqslant 1} h(x) = h(m) = q_\rho$. ∎

Numerical computations show that in general $m = 2$ seems to be not the optimal but a good parameter value, as can be seen in Table 1. In this case we get $\sqrt{M(\rho+1)} \leqslant q_\rho$. Moreover, this inequality shows that the bound $(\rho+1)/(\rho-1)$ of Theorem 1.1 is of interest only for sufficiently small values $\rho > 1$, because

$$q_{k,\rho}(f) \geqslant q_\rho \geqslant \sqrt{M(\rho+1)} > \frac{\rho+1}{\rho-1}$$

if ρ is large enough. In fact, numerical computations show that $q_\rho > (\rho+1)/(\rho-1)$ holds for $\rho \geqslant 4.42435$ such that in this case q_ρ is a better lower bound for $q_{k,\rho}(f)$ than $(\rho+1)/(\rho-1)$.

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